*Journal of Statistical Physics, Vol. 98, Nos. 1/2, 2000*

# Algorithmical Unsolvability of the Ergodicity Problem for Locally Interacting Processes with Continuous Time

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*Received March 8, 1999; final August 16, 1999*

We prove algorithmical unsolvability of the ergodicity problem for a class of one-dimensional translation-invariant random processes with local interaction with continuous time, also known as interacting particle systems. The set of states of every component is finite, the interaction occurs only between nearest neighbors, only one particle can change its state at a time and all rates are 0 or 1.

**KEY WORDS:** Random processes; interacting particle systems; ergodicity; unsolvability; Turing machine.

The theory of locally interacting particle systems or systems for short, as defined for the continuous time case in Liggett's well-known monograph,[(1)] has been developing roughly for thirty years, but the only results about algorithmical unsolvability of any problem related to them published till now pertain to the discrete time case. This note presents a proof of algorithmical unsolvability of the ergodicity problem for a class of systems with continuous time. Actually we apply the method first proposed by Kurdyumov for the discrete time case [2, 3, 4, Chap. 14] to a class of systems with continuous time with minimal necessary changes. In fact we build continuous-time systems that imitate functioning of discrete-time ones in the well-known spirit represented, e.g., by ref. 5.

Our proof is by contradiction. For every Turing Machine $M$ of a large enough class we construct a system, belonging to our class, which is ergodic if and only if $M$ stops. Thus existence of an algorithm to decide

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0022-4715/00/0100-0495$18.00/0 

which systems of our class are ergodic implies decidability of the halting problem for Turing machines, which is well-known to be false. Since the results about algorithmical unsolvability are the stronger the more narrow is the class of objects, the arbitrary finite set $S$ of states of every component is the only source of infiniteness of our class of systems, everything else is minimized: the interaction occurs only between nearest neighbors, only one particle can change its state at a time and all rates are 0 or 1.

The configuration space of a generic system of our class is $S^Z$, where $S$ is a non-empty finite set and $Z$ is the set of integer numbers. Thus a configuration is a sequence ..., $x_{-1}, x_0, x_1,...$, infinite in both directions, where $x_i \in S$ is interpreted as the state of the $i$th particle, $i \in Z$. A particle's rate of change depends only on this particle's state and the state of at most one of its nearest neighbors. Thus, for all $x, y, z \in S$, where $x \neq y$, there are rates $R_{\text{left}}(x \mid y, z)$, $R_{\text{right}}(x \mid y, z)$ and $R_{\text{center}}(x \mid y)$, all of which equal 0 or 1. During a small time $\Delta t$ three kinds of events may happen: (a) the $i$th particle changes its state from $y$ to $x$ with a probability $R_{\text{left}}(x \mid y, z) \cdot \Delta t + o(\Delta t)$ if the $(i-1)$th particle is in the state $z$, (b) the $i$th particle changes its state from $y$ to $x$ with a probability $R_{\text{right}}(x \mid y, z) \cdot \Delta t + o(\Delta t)$ if the $(i+1)$th particle is in the state $z$ and c) the $i$th particle changes its state from $y$ to $z$ with a probability $R_{\text{center}}(x \mid y) \cdot \Delta t + o(\Delta t)$. (The latter case is redundant, but convenient.)

Every process of this class has at least one invariant measure and is called ergodic if, starting from any initial measure, it tends to one and the same invariant measure. Of course, speaking about our class of processes, set of Turing machines and other classes of objects, we assume whenever necessary that each of them is enumerated in some constructive way. The following is out main result.

**Theorem.** There is no algorithm to decide for all the processes of this class, which of them are ergodic and which are not.

To prove our theorem we shall use the following set of Turing machines. Each machine has one head and one tape, which is infinite in both directions. To describe a generic Turing machine, we need $\{g_0,..., g_p\}$, the set of tape symbols and $\{h_0,..., h_q, stop\}$, the set of head states, and three functions:

$$F_{\text{tape}}: \{g_0,..., g_p\} \times \{h_0,..., h_q\} \to \{g_1,..., g_p\}$$

$$F_{\text{head}}: \{g_0,..., g_p\} \times \{h_0,..., h_q\} \to \{h_1,..., h_q, stop\}$$

$$F_{\text{move}}: \{g_0,..., g_p\} \times \{h_0,..., h_q\} \to \{-1, 0, 1\}$$

When the machine starts, the head is in the initial state $h_0$ and all cells of the tape are filled with the initial symbol $g_0$. At every step the head simultaneously writes into that cell of the tape, where it is, a new symbol according to the function $F_{\text{tape}}$, goes to a new state according to the function $F_{\text{head}}$, and moves one cell left or does not move or moves one cell right along the tape according to the values $-1, 0, 1$ of the function $F_{\text{move}}$ respectively, the arguments of all the three functions being the symbol in the presently observed cell of the tape and the present state of the head. The machine stops when and if the head reaches the state *stop*. It is well-known that the problem of deciding for all of these machines, which of them ever stop, is algorithmically unsolvable.

Now for any Turing machine $M$ of this class we shall construct a process belonging to our class. We set

$$S = S_{\text{left}} \times S_{\text{right}} \times S_{\text{age}} \times S_{\text{tape}} \times S_{\text{head}} \tag{1}$$

where

$$S_{\text{left}} = S_{\text{right}} = \{0, 1\}, \qquad S_{\text{age}} = \{-1, 0, 1\},$$

$$S_{\text{tape}} = \{g_0, \dots, g_p\}, \qquad S_{\text{head}} = \{0, h_0, \dots, h_q, stop\}.$$

Accordingly, we write a generic element of $S$ as

$$x = (left(x), right(x), age(x), tape(x), head(x)). \tag{2}$$

We say that a state $x$ and a particle in this state has a left bracket if $left(x) = 1$ and that it has a right bracket if $right(x) = 1$. We call $x$ a no-head if $head(x) = 0$ and a head otherwise. We call a head $x$ an adult if $age(x) = 0$ and a child otherwise. We call an adult $x$ a stop-adult if $head(x) = stop$. A child $x$ is either a left-child if $age(x) = -1$ or a right-child if $age(x) = 1$. We say that an adult $x$ wants to move left, to stay or to move right when $F_{\text{move}}(tape(x), head(x))$ equals $-1$, 0 or 1 respectively. The state $(0, 0, 0, g_0, stop)$ is called *final*.

Our proof is based on the following ideas. The functioning of our system is similar to simultaneous functioning of many representations of the head of the original Turing machine $M$ on one and the same tape and along with defining our rates we shall explain how they imitate this functioning. Since only one particle can change its state at a time, a representation of the head of $M$ in our system cannot just jump from one site to another. That is why we need heads of three types: adult, left-child and right-child. A generic configuration of our system contains infinitely many representations of the head of $M$, which evolve in time imitating its functioning. Each representation occupies one site or two adjacent sites. Each representation belongs to its area, which is a segment of $Z$ filled with tape symbols written by this representation. It is important that no representation ever reads a symbol written by another representation.

Now let us define our rates by listing those cases when they equal 1. These cases are called flips, enumerated and commented. In all the other cases our rates are zeros. Some flips come in symmetric pairs, each pair consisting of the left and right version. We formulate only one version in each pair, the other version to be obtained from the given one by permuting left and right. Thus the only changes to occur are the following ones and their symmetric versions. All the definitions of flips assume that $x \neq y$. All of them except flips 7, 8-left and 8-right assume that none of $x$, $y$, $z$ is a final and none of $y$, $z$ is a stop-adult.

*Flip 1.* Suppose that $x = (1, 1, 0, g_0, h_0)$. Then $R_{\text{center}}(x \mid y) = 1$. *Comment*: An adult in the initial state with the initial tape symbol and both brackets is born. Thus the state $(1, 1, , g_0, h_0)$, which imitates the initial state of $M$, can appear anywhere with the exceptions specified above.

As soon as the initial state of $M$ is created, it needs to function, and this is imitated in our system by flips 2–6.

*Flip 2.* Suppose that

$$x = (\mathit{left}(y), \mathit{right}(y), 0, F_{\text{tape}}(\mathit{tape}(y), \mathit{head}(y)), F_{\text{head}}(\mathit{tape}(y), \mathit{head}(y)))$$

and $y$ is an adult, which wants to stay. Then $R_{\text{center}}(x \mid y) = 1$. *Comment*: If an adult wants to stay, it simultaneously updates the tape symbol and head state.

The following flips imitate the movements of the head of $M$.

*Flip 3-left.* Suppose that

$$x = (0, 1, 1, g_0, F_{\text{head}}(\mathit{tape}(z), \mathit{head}(z)))$$

and $z$ is an adult, which wants to move right and has a right bracket. Then $R_{\text{left}}(x \mid y, z) = 1$. *Comment*: If an adult wants to move right and has a right bracket, then a right-child is created on its right side with these initial tape symbol, with head state determined by the function $F_{\text{head}}$, with right bracket and without left bracket.

*Flip 4-left.* Suppose that

$$x = (0, \mathit{right}(y), 1, \mathit{tape}(y), F_{\text{head}}(\mathit{tape}(z), \mathit{head}(z))),$$

that $y$ has no left bracket and $z$ is an adult, which wants to move right and has no right bracket. Then $R_{\text{left}}(x \mid y, z) = 1$. *Comment*: If an adult wants to move right and has no right bracket and its right neighbor has no left bracket, then a right-child is created there, with head state determined by the function $F_{\text{head}}$, but the tape symbol and brackets do not change there.

Let us explain the difference between flips 3-left and 4-left. Both start the sequence of flips imitating one move of the head of $M$ in the right direction. If our adult $z$ is at the right end of its area, which is recognized by its having a right bracket, flip 3-left may occur, otherwise flip 4-left may occur. If flip 3-left occurs, the site on its right side gets included into its area, and excluded from another area if it belonged there. If flip 4-left occurs, no area is changed.

*Flip 5-right.* Suppose that

$$x = (\mathit{left}(y), 0, 0, F_{\text{tape}}(\mathit{tape}(y), \mathit{head}(y)), 0)),$$

that $y$ is an adult, which wants to move right, that $z$ is a right-child, which has no left bracket and that $\mathit{head}(z) = F_{\text{head}}(\mathit{tape}(y), \mathit{head}(y))$. Then $R_{\text{right}}(x \mid y, z) = 1$. *Comment*: If an adult has a right-child on its right side, it dies, its right bracket (if present) disappears and the tape symbol is updated. Flip 5-right is the second flip in the sequence imitating one move of the head of $M$. It may occur when an adult has created a right-child on the right side of it and must die to let it become adult.

However, flip 5-right may occur in a different situation also, namely when that adult which created this right-child has been replaced by another adult of a different representation (e.g., a new-born). In this case a right-child "by mistake" kills an adult which is not its father. However, this may happen only if the adult being killed has a right bracket and wants to move right and to create exactly the same right-child. In this case we classify the child as belonging to the same representation as that adult which is being killed.

*Flip 6-left.* Suppose that $x$ coincides with $y$ except $\mathit{age}(x) = 0$ and $\mathit{age}(y) = 1$. Suppose also that $z$ is not a head. Then $R_{\text{left}}(x \mid y, z) = 1$. *Comment*: If a right-child sees no head on its left side, it becomes adult.

Thus one move of the head of $M$ in the right direction is imitated by a sequence of three flips: first either flip 3-left or flip 4-left, second flip 5-right, and third flip 6-left. One move of the head of $M$ in the left direction is imitated in a symmetric way.

*Flip 7.* Suppose that $x$ is final and $y$ is a non-final stop-adult. Then $R_{\text{center}}(x \mid y) = 1$.

*Flip 8-left.* Suppose that $x$ and $z$ are final. Then $R_{\text{left}}(x \mid y, z) = 1$. *Comment*: The flip 7 assures that as soon as some head reaches a stop-adult state, it turns into the final state, which expands in both directions due to flips 8-left and 8-right.

Our process is defined. Our theorem immediately follows from the fact that this process is ergodic if and only if $M$ stops. To prove this, let us denote $\delta_{\text{final}}$ the measure concentrated in the configuration "all the particles are in the final state." Since $\delta_{\text{final}}$ is invariant, our process is ergodic if and only if it tends to $\delta_{\text{final}}$ from any initial configuration. Thus it is sufficient to argue in the following two directions.

*One Direction.* Suppose that $M$ stops. Then for any initial configuration the following scenario is possible: First, an adult is born due to flip 1 and no other flip 1 occurs in a large enough vicinity during long enough time. As time goes on, this representation of the head of $M$ evolves due to flips 2–6. Since $M$ stops, this functioning eventually produces a stop-adult. Due to flips 7, 8-left and 8-right this stop-adult turns into the final state, which expands in both directions, whereby the measure tends to $\delta_{\text{final}}$. Since the probability of this scenario to happen at any particular region is positive, it happens somewhere almost sure.

*The Other Direction.* Let us assume that $M$ never stops, take the initial configuration "all particles are in the initial state," where the initial state is $(0, 0, 0, g_0, 0)$, and prove that the probability of a particle to be in tide final state remains zero all the time. Let us denote $x(s, t) \in S$ the state of the $s$th particle at time $t$. Let us assume that $x(s_0, t_0) = \textit{final}$ with a positive probability and come to a contradiction. For this purpose we cover the event $x(s_0, t_0) = \textit{final}$ by a countable set of events (except a set of zero measure) and prove that everyone of them contradicts out assumption. First we go back from the point $(s_0, t_0)$ and in result represent our event as a consequence of a finite configuration of flips. By the way we recognize all the flips (i.e., their numbers and left-rightness) of that configuration. In particular, we always can distinguish flip 1 from other numbers due to the fact that $F_{\text{tape}}(\cdot)$ never equals $g_0$ and $F_{\text{head}}(\cdot)$ never equals $h_0$.

Then starting from $t = 0$ and increasing $t$, we can distinguish by induction, which heads correspond to one and the same representation of the head of $M$. Also by induction we can distinguish this representation's area, that is the set of those sites, which have tape symbols ever written by it. Also we prove by induction that at any time this area includes exactly one or two heads, namely an adult or a child or an adult and a child generated by it and stretches from them to the left until there is a left bracket inclusively or a right bracket exclusively. In the right direction it stretches according to a symmetric rule.

Our flips are defined in such a way that no representation ever reads a symbol written by another one. In particular, when flip 3-left creates a right-child, its tape symbol becomes initial, so that the former symbol is forgotten. Flip 4-left allows an adult to create a right-child only on condition that that place had no left bracket, which means that it already belonged to the area of that adult. Thus we can prove by induction that for any area and any time the configuration at that area at that time coincides with the configuration on some part of the tape in the process of functioning of $M$. Therefore the head states that ever appear in our process with positive probabilities are only those which appear in the course of functioning of $M$. Since $M$ never *stops*, these states do not include stop. Thus we get the contradiction we sought.

## ACKNOWLEDGMENT

This work was supported by FAPESP, Grant No. 98/15994-0.

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*Communicated by J. L. Lebowitz*